Eigenvalues and Eigenvectors - Problems 2

For each of the following matrices A, find the eigenvalues λ of A and for each λ determine a maximal set of linearly independent eigenvectors associated to λ. Say then if the matrix is diagonalizable or not, and motivate your answer. In the case A is diagonalizable, determine an invertible matrix P such that $P^{-1}AP = D$ is diagonal.

1. $$A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix}$$

2. $$A = \begin{pmatrix} 6 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

3. $$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Eigenvalues and Eigenvectors - Answers 2

1. The characteristic equation of A is $\det(A-\lambda I)=(2-\lambda)(\lambda^2-4\lambda+3)=0$, so the eigenvalues are $\lambda=1,2$ and 3. Each eigenvalue has only one linearly independent eigenvector. The eigenvalue $\lambda=1$ has $\begin{pmatrix}1\\4\\-1\end{pmatrix}$ as an eigenvector, while $\lambda=2$ has $\begin{pmatrix}0\\-1\\0\end{pmatrix}$ and $\lambda=3$ has $\begin{pmatrix}1\\2\\1\end{pmatrix}$.
   Since there are three linearly independent eigenvectors, A is diagonalizable.
   The matrix P is constructed by taking the eigenvectors and making them the columns of P. Then
$$P^{-1}AP=\begin{pmatrix}1&0&1\\4&-1&2\\-1&0&1\end{pmatrix}^{-1}\begin{pmatrix}2&0&1\\-1&2&3\\1&0&2\end{pmatrix}\begin{pmatrix}1&0&1\\4&-1&2\\-1&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix}=D.$$

2. The characteristic equation of A is $\det(A-\lambda I)=(1+\lambda)(\lambda^2-9\lambda+14)=0$, so the eigenvalues are $\lambda=-1,2$ and 7. Each eigenvalue has only one linearly independent eigenvector. The eigenvalue $\lambda=-1$ has $\begin{pmatrix}0\\0\\1\end{pmatrix}$ as an eigenvector, while $\lambda=2$ has $\begin{pmatrix}1\\-2\\0\end{pmatrix}$ and $\lambda=7$ has $\begin{pmatrix}2\\1\\0\end{pmatrix}$.
   Since there are three linearly independent eigenvectors, A is diagonalizable.
   The matrix P is constructed by taking the eigenvectors and making them the columns of P. Then
$$P^{-1}AP=\begin{pmatrix}0&1&2\\0&-2&1\\1&0&0\end{pmatrix}^{-1}\begin{pmatrix}6&2&0\\2&3&0\\0&0&-1\end{pmatrix}\begin{pmatrix}0&1&2\\0&-2&1\\1&0&0\end{pmatrix}=\begin{pmatrix}-1&0&0\\0&2&0\\0&0&7\end{pmatrix}=D.$$

3. The characteristic equation of A is $\det(A-\lambda I)=\lambda(2-\lambda)(\lambda-3)=0$, so the eigenvalues are $\lambda=0,2$ and 3. Each eigenvalue has only one linearly independent eigenvector. The eigenvalue $\lambda=0$ has $\begin{pmatrix}1\\1\\-2\end{pmatrix}$ as an eigenvector, while $\lambda=2$ has $\begin{pmatrix}1\\-1\\0\end{pmatrix}$ and $\lambda=3$ has $\begin{pmatrix}1\\1\\1\end{pmatrix}$.
   Since there are three linearly independent eigenvectors, A is diagonalizable.
   The matrix P is again constructed by taking the eigenvectors and making them the columns of P. Then
$$P^{-1}AP=\begin{pmatrix}1&1&1\\1&-1&1\\-2&0&1\end{pmatrix}^{-1}\begin{pmatrix}2&0&1\\0&2&1\\1&1&1\end{pmatrix}\begin{pmatrix}1&1&1\\1&-1&1\\-2&0&1\end{pmatrix}=\begin{pmatrix}0&0&0\\0&2&0\\0&0&3\end{pmatrix}=D.$$